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Intensity distribution for waves in disordered media: Deviations from Rayleigh statistics

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We study the intensity distribution function $P(I)$ for monochromatic waves propagating in a quasi-one-dimensional disordered medium, assuming that a point source and a point detector are embedded in the bulk of the medium. We find deviations from the Rayleigh statistics at moderately large I and a logarithmically normal asymptotic behavior of $P(I)$. When the radiation source and the detector are located close to the opposite edges of the sample (on a distance much less then the sample length), an intermediate regime with a stretched exponential behavior of $P(I)$ emerges. [S1063-651X(98)51306-2]

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When a wave propagates through a random medium, it undergoes multiple scattering from inhomogeneities. The scattered intensity pattern (speckle pattern) is highly irregular and should be described in statistical terms. One of its characteristics is the intensity distribution $P(I)$ at some point $\mathbf{r}$. Almost a century ago Lord Rayleigh, using simple statistical arguments, proposed a distribution that bears his name:

$$P_0(\tilde{I})=\exp(-\tilde{I}), \tag{1}$$

where $\tilde{I}$ is the intensity normalized to its average value, $\tilde{I}=I/\langle I\rangle$. The Rayleigh distribution has moments $\langle \tilde{I}^n\rangle = n!$ and it provides, in many cases, a rather accurate fit to experimental data, as long as I is not too large [1]. For large I, however, the data show large deviations from Eq. (1) [1–3]. Various extensions of Eq. (1) have been proposed in the literature. Jakeman and Pusey [4] proposed to fit the data with the K distribution. It contains a phenomenological parameter η and its moments are given by $\langle \tilde{I}^n\rangle = n!\,\eta^{-n}\Gamma(n+\eta)/\Gamma(\eta)$. The experimentally relevant situation corresponds to $\eta \gg 1$. In this case moments up to $n \lesssim \eta$ can be approximated as

$$\langle \tilde{I}^n\rangle \simeq n!\exp(n^2/\eta), \tag{2}$$

where only the leading term in the exponent has been kept. Thus, only low moments ($n \ll \sqrt{\eta}$) are close to the Rayleigh value $n!$. Some theoretical support to the phenomenological formula Eq. (2) has been given by Dashen [5], who considered smooth disorder (the typical size of inhomogeneities is much larger than the wavelength).

More recently, there has been a considerable amount of theoretical study of the statistics of the transmission coefficients t_{ab} of a one-dimensional sample with short-range disorder [6–9]. In this formulation of the problem, a source and a detector of the radiation are located outside the sample. The source produces a plane wave injected into an incoming channel a, and the intensity in an outgoing channel b is measured. It was shown in [7–9] that the distribution of the normalized transmission coefficients $s_{ab}=t_{ab}/\langle t_{ab}\rangle$ crosses over from the Rayleigh distribution $P(s_{ab})=e^{-s_{ab}}$ to a stretched-exponential one $P(s_{ab})\sim e^{-2\sqrt{g s_{ab}}}$, where g is the dimensionless conductance.

In this paper we consider a different situation, where both the source and the detector of the radiation are embedded in the bulk of the sample, and we calculate the intensity distribution $P(I)$ in this case. We prove that, for not too large n, the moments can indeed be described by Eq. (2), and we compute the parameter η phenomenologically introduced in [4]. We further compute the whole distribution function $P(I)$ and show that its asymptotic behavior at large I is of a logarithmically normal form, in contrast to the stretched-exponential asymptotics of $P(t_{ab})$ found in Refs. [7–9]. Finally, we discuss how these two different forms of the asymptotic behavior match each other and describe physical mechanisms governing both of them.

We assume a quasi-one-dimensional (1D) geometry, i.e., we consider a tube of transverse dimension W and length $L \gg W$, filled with scattering medium (Fig. 1). The monochromatic source of radiation is placed at point $\mathbf{r}_0$. The field at some point $\mathbf{r}$ is given by the (retarded) Green's function $G_R(\mathbf{r},\mathbf{r}_0)$ and the radiation intensity is defined as $I(\mathbf{r},\mathbf{r}_0)\equiv|G_R(\mathbf{r},\mathbf{r}_0)|^2$. The average intensity $\langle I(\mathbf{r},\mathbf{r}_0)\rangle$ is represented diagrammatically in Fig. 2. It consists of a diffusion ladder (a diffuson) $T(\mathbf{r}_1,\mathbf{r}_2)$ attached to two external vertices. The vertices are short-range objects and can be approximated by a δ function times $(\ell/4\pi)$, so that $\langle I(\mathbf{r},\mathbf{r}_0)\rangle=(\ell/4\pi)^2 T(\mathbf{r},\mathbf{r}_0)$. For the quasi-one-dimensional geometry, the expression for the diffuson reads

$$T(\mathbf{r},\mathbf{r}_0)=\left(\frac{4\pi}{\ell}\right)^2\frac{3}{4\pi}\,\frac{[z_<(L-z_>)]}{A\ell L}, \tag{3}$$

where ℓ is the elastic mean free path, A is the cross section of the tube, the z axis is directed along the sample, and $z_<=\min(z,z_0)$ and $z_>=\max(z,z_0)$. We assume, of course, that $|z-z_0|\gg W$.

The intensity distribution $P(I)$, in the diagrammatic approach, is obtained by calculating the moments $\langle I^n\rangle$ of the intensity. In the leading approximation [10], one should draw n-retarded and n-advanced Green's functions and insert ladders between pairs $\{G_R, G_A\}$ in all possible ways. This leads to $\langle I^n\rangle = n!\langle I\rangle^n$ and, thus, to Eq. (1).

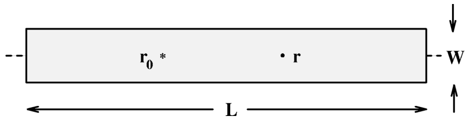

FIG. 1. Geometry of the problem. Points $\mathbf{r}_0=(x_0,y_0,z_0)$ and $\mathbf{r}=(x,y,z)$ are the positions of the source and the observation point, respectively.

Corrections to the Rayleigh result come from diagrams with intersecting ladders, which describe interaction between diffusons. The leading correction is due to pairwise interactions. The diagram in Fig. 3 represents a pair of ‘‘colliding’’ diffusons. The algebraic expression for this diagram is

$$C(\mathbf{r},\mathbf{r}_0)=2\left(\frac{\ell}{4\pi}\right)^4\int\left(\prod_{\iota=1}^4 d^3\mathbf{r}_\iota\right)\times T(\mathbf{r},\mathbf{r}_1)T(\mathbf{r},\mathbf{r}_2)T(\mathbf{r}_3,\mathbf{r}_0)T(\mathbf{r}_4,\mathbf{r}_0)\times\left\{\left(\frac{\ell^5}{48\pi k_0^2}\right)\int d^3\boldsymbol{\rho}[(\nabla_1+\nabla_2)\cdot(\nabla_3+\nabla_4)+2(\nabla_1\cdot\nabla_2)+2(\nabla_3\cdot\nabla_4)]\prod_{\iota=1}^4\delta(\boldsymbol{\rho}-\mathbf{r}_\iota)\right\}, \quad (4)$$

where k_0 is the wave number and ∇_ι acts on $\mathbf{r}_\iota$. The factor $(\ell/4\pi)^4$ comes from the four external vertices of the diagram, the T's represent the two incoming and two outgoing diffusons, and the expression in the curly brackets corresponds to the internal (interaction) vertex [11]. Finally, the factor 2 accounts for the two possibilities of inserting a pair of ladders between the outgoing Green's functions. Integrating by parts and employing the quasi-one-dimensional geometry of the problem, we obtain (for $z_0<z$)

$$C(z,z_0)\simeq 2\langle I(z,z_0)\rangle^2\left(1+\frac{4}{3\gamma}\right), \quad (5)$$

where $\langle I(z,z_0)\rangle=(3/4\pi)[z_0(L-z)/A\ell L]$ is the average intensity,

$$\gamma=2g\frac{L^3}{L^2(3z+z_0)-2Lz(z+z_0)+2z_0^2(z-z_0)}\gg 1, \quad (6)$$

and $g=k_0^2\ell A/3\pi L\gg 1$ is the dimensionless conductance of the tube. For simplicity, we will assume that the source and the detector are located relatively close to each other, so that $|z-z_0|\ll L$, in which case Eq. (6) reduces to $\gamma=gL^2/2z(L-z)$. (All the results are found to be qualitatively the same in the generic situation $z_0\sim z-z_0\sim L-z\sim L$.)

In order to calculate $\langle I^n\rangle$ one has to compute a combinatorial factor that counts the number N_i of diagrams with i pairs of interacting diffusons. This number is [6] $N_i=(n!)^2/[2^{2i}i!(n-2i)!]\simeq(n!/i!)(n/2)^{2i}$, so that

$$\frac{\langle I^n\rangle}{\langle I\rangle^n}=n!\sum_{\iota=0}^{[n/2]}\frac{1}{i!}\left(\frac{2n^2}{3\gamma}\right)^i\simeq n!\exp(2n^2/3\gamma). \quad (7)$$

Although i cannot exceed $n/2$, the sum in Eq. (7) can be extended to ∞, if the value of n is restricted by the condition $n\ll\gamma$. Equation (7) represents the leading exponential correction to the Rayleigh distribution. Let us now discuss the effect of higher-order ‘‘interactions’’ of diffusons. Diagrams with three intersecting diffusons will contribute a correction of n^3/γ^2 in the exponent of Eq. (7), which is small compared to the leading correction in the whole region $n\ll\gamma$, but becomes larger than unity for $n\gtrsim\gamma^{2/3}$. Likewise, diagrams with four intersecting diffusons produce a n^4/γ^3 correction, etc. Restoring the distribution $P(I)$, we find that

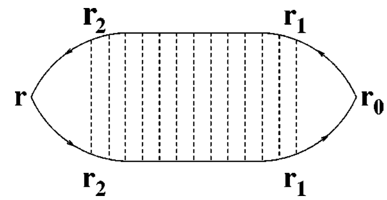

FIG. 2. Diagram for the average intensity. The diffusion ladder is inserted between two solid lines that represent the average Green's functions.

$$P(\tilde{I})\simeq\exp\left[-\tilde{I}+\frac{2}{3\gamma}\tilde{I}^2+O\left(\frac{\tilde{I}^3}{\gamma^2}\right)+\cdots\right]. \quad (8)$$

It should be realized that Eq. (8) is applicable only for $\tilde{I}\ll\gamma\sim g$ and thus does not determine the far asymptotics of $P(I)$. The latter is inaccessible by the perturbative diagram technique and is handled below by the supersymmetry method.

In the supersymmetry formalism, averaging over disorder is replaced by functional integration over supermatrix fields $Q(\mathbf{r})$ that satisfy the constraint $Q^2=1$ [12]. For technical simplicity, we will assume that the time reversal symmetry is broken by some magneto-optical effects. The integration is done with a weight function $\exp[-S\{Q\}]$, where $S\{Q\}$ is the σ-model action,

$$S\{Q\}=-\frac{\pi\nu D}{4}\int d^3\mathbf{r}\ \mathrm{tr}_S(\nabla Q)^2, \quad (9)$$

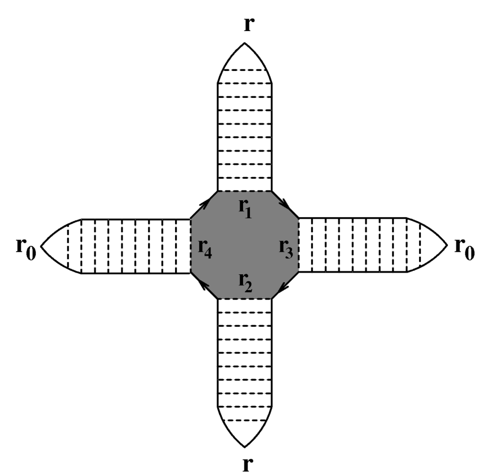

FIG. 3. Diagram for a pair of interacting diffusons. The external vertices contribute the factor $(\ell/4\pi)^4$. The shaded region denotes the internal interaction vertex [see Eq. (4)].

tr_S denotes the supertrace, D is the diffusion constant, and ν is the average density of states. In the considered quasi-1D geometry, $\pi\nu D = gL/2A$, and the field Q depends on the z coordinate only, yielding $S\{Q\} = -(gL/8)\int dz\ \mathrm{tr}_S(dQ/dz)^2$. Following the derivation outlined in [13–15], the moments of the intensity at point $\mathbf{r}$ due to the source at $\mathbf{r}_0$ are given by

$$\langle I^n\rangle = \left(-\frac{k_0^2}{16\pi^2}\right)^n \int [\mathcal{D}Q][Q_{12}^{bb}(z)]^n[Q_{21}^{bb}(z_0)]^n e^{-S\{Q\}}, \tag{10}$$

where $Q_{12}^{bb}(Q_{21}^{bb})$ is the retarded-advanced (respectively, advanced-retarded) matrix element in the boson-boson sector of Q. Assuming again that the two points $\mathbf{r}$ and $\mathbf{r}_0$ are sufficiently close to each other, $|z-z_0| \ll L$, and taking into account the slow variation of the Q field along the sample, we can replace the product $Q_{12}^{bb}(z)Q_{21}^{bb}(z_0)$ by $Q_{12}^{bb}(z)Q_{21}^{bb}(z)$. We then get the following result for the distribution of the dimensionless intensity $y = (16\pi^2/k_0^2)I$:

$$P(y) = \int dQ\ \delta(y + Q_{12}^{bb}Q_{21}^{bb})Y(Q), \tag{11}$$

where $Y(Q)$ is a function of a single supermatrix Q, which is defined as follows [13,15]:

$$Y(Q_0) \equiv \int_{Q(\mathbf{r}_0)=Q_0} [\mathcal{D}Q]\exp[-S\{Q\}]. \tag{12}$$

In general, the function $Y(Q)$ depends only on the parameters $1 \leq \lambda_1 < \infty$, $-1 \leq \lambda_2 \leq 1$ entering into the standard parametrization of the Q matrices [16]. Performing the integration over the other degrees of freedom, we find that

$$P(y) = \left(\frac{d}{dy} + y\frac{d^2}{dy^2}\right)\int d\lambda_1 d\lambda_2 \times\left(\frac{\lambda_1+\lambda_2}{\lambda_1-\lambda_2}\right)Y(\lambda_1,\lambda_2)\,\delta(y+1-\lambda_1^2). \tag{13}$$

The evaluation of $Y(Q_0) = Y(\lambda_1^0,\lambda_2^0)$ involves, by its definition (12), an integration over all supermatrix fields, which assume a given value Q_0 at point z_0 and satisfy the boundary conditions $Q|_{z=0,L} = \Lambda$, where $\Lambda \equiv \mathrm{diag}\{1,1,-1,-1\}$. Since $g \gg 1$, this calculation can be done by the saddle-point method, as suggested by Muzykantskii and Khmelnitskii [17]. The result is [15]

$$Y(\lambda_1,\lambda_2) \simeq \exp\left\{-\frac{\gamma}{2}[\theta_1^2+\theta_2^2]\right\}, \tag{14}$$

where $\lambda_1 \equiv \cosh\theta_1$, $\lambda_2 \equiv \cos\theta_2$ $(0 \leq \theta_1 < \infty,\ 0 \leq \theta_2 \leq \pi)$. In fact, the dependence of Y on θ_2 is not important, within the exponential accuracy, because it simply gives a prefactor after the integration in Eq. (13). Therefore, up to a pre-exponential factor, the distribution function $P(y)$ is given by

$$P(y) \sim Y(\lambda_1 = \sqrt{1+y}, \lambda_2 = 1) \sim \exp(-\gamma\theta_1^2/2), \tag{15}$$

where $\theta_1 = \ln(\sqrt{1+y} + \sqrt{y})$. Finally, after normalizing y to its average value $\langle y\rangle = 2/\gamma$, we obtain

$$P(\tilde{I}) \simeq \exp\left\{-\frac{\gamma}{2}[\ln^2(\sqrt{1+2\tilde{I}/\gamma} + \sqrt{2\tilde{I}/\gamma})]\right\}. \tag{16}$$

For $\tilde{I} \ll \gamma$, Eq. (16) reproduces the perturbative expansion (8), while for $\tilde{I} \gg \gamma$ it implies the log-normal asymptotic behavior of the distribution $P(\tilde{I})$:

$$\ln P(\tilde{I}) \simeq -(\gamma/8)\ln^2(8\tilde{I}/\gamma). \tag{17}$$

The log-normal ''tail'' (17) should be contrasted with the stretched-exponential asymptotic behavior of the distribution of transmission coefficients [7–9]. Let us briefly discuss how these two results match each other. Analyzing the expression for the moments (10), we find that when the points z and z_0 approach the sample edges, $z_0 = L - z \ll L$, an intermediate regime of stretched-exponential behavior emerges:

$$\ln P(\tilde{I}) \simeq \begin{cases} -\tilde{I} + \dfrac{1}{3g}\tilde{I}^2 + \cdots, & \tilde{I} \ll g \\ -2\sqrt{g\tilde{I}}, & g \ll \tilde{I} \ll g\left(\dfrac{L}{z_0}\right)^2 \\ -\dfrac{gL}{8z_0}\ln^2\left[16\left(\dfrac{z_0}{L}\right)^2\dfrac{\tilde{I}}{g}\right], & \tilde{I} \gg g\left(\dfrac{L}{z_0}\right)^2. \end{cases} \tag{18}$$

Thus, when the source and the detector move toward the sample edges, the region of validity of the stretched-exponential behavior becomes broader, while the log-normal tail gets pushed further away. In contrast, when the source and the detector are located deep in the bulk, $z_0 \sim L - z \sim L$, the stretched-exponential regime disappears, and the Rayleigh distribution crosses over directly to the log-normal one at $\tilde{I} \sim g$.

Let us now describe the physical mechanisms behind these different forms of $P(\tilde{I})$. The Green's function $G^R(\mathbf{r_0},\mathbf{r})$ can be expanded in eigenfunctions of a non-Hermitian (due to open boundaries) ''Hamiltonian'' as $G^R(\mathbf{r_0},\mathbf{r}) = \Sigma_i \psi_i^*(\mathbf{r_0})\psi_i(\mathbf{r})(k_0^2 - E_i + i\gamma_i)^{-1}$. Since the level widths γ_i are typically of the order of the Thouless energy $E_c \sim D/L^2$, there is typically $\sim g$ levels contributing appreciably to the sum. In view of the random phases of the wave functions, this leads to a Gaussian distribution of $G^R(\mathbf{r_0},\mathbf{r})$ with zero mean, and thus to the Rayleigh distribution of $I(\mathbf{r_0},\mathbf{r}) = |G^R(\mathbf{r_0},\mathbf{r})|^2$, with the moments $\langle\tilde{I}^n\rangle = n!$. The stretched-exponential behavior results from the disorder realizations, where one of the states ψ_i has large amplitudes in both points $\mathbf{r_0}$ and $\mathbf{r}$. Considering both $\psi_i(\mathbf{r_0})$ and $\psi_i(\mathbf{r})$ as independent random variables with Gaussian distribution, and taking into account that only one (out of g) term contributes in this case to the sum for G^R, we find that $\langle\tilde{I}^n\rangle \sim n!n!/g^n$, which corresponds to the above stretched-exponential form of $P(\tilde{I})$. Finally, the log-normal asymptotic behavior corresponds to those disorder realizations, where G^R is dominated by an anomalously localized state, which have an atypically small width γ_i (the same mechanism determines the log-normal asymptotics of the distribution of local density of states (see Refs. [15,18]).

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